Decomposition of positive projections on C^*-algebras.

by

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1. Introduction.

Suppose B is a unital C^*-algebra and that $P: B \to B$ is a unital positive projection, i.e. $P \geq 0$, $P(1) = 1$, and $P^2 = P$. It is known [4,18] that the image of P is a C^*-algebra under the product $a \cdot b = P(ab)$ if and only if P is completely positive. Thus when B acts on a Hilbert space H, $P(B)$ is a C^*-algebra if and only if there are a linear isometry V of H on a Hilbert space K and a $*$-representation π of B on K such that

$$(1.1) \qquad P(x) = V^*\pi(x)V$$

for all $x \in B$.

In a previous paper with E. Effros [8] it was shown that if the image A of the self-adjoint part B_h of B under P was given the product

$$a * b = P(a \circ b)$$

where $a \circ b = \frac{1}{2}(ab + ba)$, then A has a faithful representation as a JC-algebra, i.e. a norm closed Jordan algebra of self-adjoint operators on a Hilbert space. More concretely, if $N = \{a \in B_h : P(a^2) = 0\}$ then $A + N$ is a JC-subalgebra of B_h, and P restricted to $A + N$ is a Jordan homomorphism of $A + N$ onto A with kernel N. Since there are lots of examples when A is not the self-adjoint part of a C^*-algebra, see [8], we cannot in general expect P to be completely positive. However, we might expect that if we symmetrize the definition of π in (1.1) to be a Jordan homomorphism, i.e. by [14] a sum of a $*$-homomorphism and $*$-anti-homomorphism, then a decomposition like (1.1) might hold. In this case P is <u>decomposable</u> in the terminology of [13].

In the present paper we shall characterize those projections P which are decomposable, the characterization being in terms of the JC-algebra $A+N$. Recall that a JC-algebra is said to be <u>reversible</u> if it is closed under symmetric products $a_1a_2\ldots a_n + a_n\ldots a_2a_1$ when the a_i's lie in the algebra. While the main result introduces a technical condition called "weakly decomposable", if we add extra assumptions on A or P we avoid this and obtain the following corollary:

<u>Theorem</u>. Let B be a unital C^*-algebra and P a unital positive projection of B into itself. Let $A = P(B_h)$ and $N = \{a \in B_h : P(a^2) = 0\}$. If A is a JC-subalgebra of B_h, P is decomposable if and only if A is reversible. If the restriction of P to the C^*-algebra generated by A is faithful, then P is decomposable if and only if $A+N$ is a reversible JC-subalgebra of B_h

From the first statement in the theorem a unital positive projection onto a spin factor whose real dimension is more than six, is never decomposable, see [8] for examples. Another consequence is that if A is the set of fixed points in B_h under a family of Jordan automorphisms of B, then a positive projection onto A is automatically decomposable.

It was noted by Connes, [8, Cor.1.6], that if φ is a normal unital positive linear map of a von Neumann algebra M into itself, and $A = \{a \in M_h : \varphi(a) = a\}$, then there is a positive projection of M

into itself with $P(M_h) = A$, hence A has a faithful representation as a weakly closed JC-algebra. It is easy to see that if φ is decomposable then so is P, hence the representation is onto a reversible JC-algebra. Since A is the eigenspace in M_h for the eigenvalue 1 we have thus obtained an algebraic condition on one of the eigenspaces, which is necessary in order that φ be decomposable. This result is of interest for two reasons, firstly because it shows that a future theory of spectral subspaces of positive maps might be quite rich and useful, and secondly, because it was for a long time an open question whether all positive maps were decomposable. A counterexample was first exhibited by Choi [3], and then more examples were given by Woronowics [20,21], who termed decomposable maps as being of "jordanian type".

The proof of the main theorem, Theorem 7.1, and thus the paper itself, is divided into several sections. In paragraph 2 we prove that the set of decomposable maps is closed under the usual algebraic notions like sums, composition, etc., and also practically the easy part of the theorem, the necessity of $A + N$ being reversible. In paragraph 3 we show some general results on JC-algebras, and in paragraph 4 we show the first main step towards the proof of the theorem, namely if $P(B_h)$ is a reversible JC-subalgebra of B_h, in order to conclude that P is decomposable it suffices to consider the case when B is a von Neumann algebra and $P(B_h)$ is a JW-factor of type I. In paragraph 5 we consider this situation in the special case when B is the 4×4 or the 2×2 complex matrices, and in paragraph 6 these results are extended to general von Neumann algebras when $P(B_h)$ is a reversible JW-factor of type I. Then the proof is completed in paragraph 7.

For later references we recall the basic concepts on Jordan algebras. In the present paper a JC-algebra is a norm closed Jordan subalgebra of the self-adjoint operators in a C^*-algebra equipped with the product $a \circ b = \frac{1}{2}(ab + ba)$. A weakly closed JC-algebra A is called a JW-algebra. If moreover its center $A \cap A'$ is the scalars, A is said to be a JW-factor. It is said to be a JW-factor of type I if it is moreover generated by its minimal projections. It is then of type I_n if there exist n orthogonal minimal projections in A with sum 1. Otherwise we refer the reader to the papers on JC-algebras referred to in the bibliography.

2. Decomposable maps.

Let B be a C^*-algebra and H a (complex) Hilbert space. As in [13, Definition 7.1] we say a positive linear map φ of B into the bounded operators $B(H)$ on H is decomposable if there are a Hilbert space K, a bounded linear operator V of H into K, and a Jordan $*$-homomorphism π of B into $B(K)$ such that

$$\varphi(x) = V^*\pi(x)V \tag{2.1}$$

for all $x \in B$. Woronowicz [21] used the term of jordanian type for such maps. If φ is a positive linear map of B into $B(H)$ such that $\varphi \otimes \iota_n$ is a positive map of $B \otimes M_n$ into $B(H) \otimes M_n$ for all $n \in \mathbb{N}$, where M_n is the complex $n \times n$ matrices, and ι_n is the identity map on M_n, then φ is said to be completely positive [12], and φ is said to be completely co-positive [21] if $\varphi \otimes t_n$ is a positive map of $B \otimes M_n$ into $B(H) \otimes M_n$ for all n, where t_n is the transpose map on M_n. The celebrated Neumark-Stinespring theorem [12] states that a map φ is completely positive if and only if φ is decomposable with π in (2.1) being a $*$-homomorphism. Analogously φ is completely co-positive if and only if φ is decomposable with π a $*$-anti-homomorphism.

We shall in the present section show the basic elementary properties of decomposable maps.

Lemma 2.1. Let B be a C^*-algebra and H a Hilbert space. Then a linear map φ of B into $B(H)$ is decomposable if and only if φ is the sum of a completely positive and a completely co-positive map of B into $B(H)$.

Proof. By [14] every Jordan $*$-homomorphism from B into $B(K)$, where K is a Hilbert space, is the sum of a $*$-homomorphism and a $*$-anti-homomorphism. Hence it is immediate that a decomposable map is the sum of a completely positive and a completely co-positive map.

Conversely assume $\varphi = \varphi_1 + \varphi_2$ with φ_1 completely positive and φ_2 completely co-positive maps of B into $B(H)$. Say φ_i has a decomposition $\varphi_i = V_i^* \pi_i V_i$ with $V_i : H \to K_i$ bounded linear operators, and π_1 a $*$-homomorphism and π_2 a $*$-anti-homomorphism of B into $B(K_1)$ and $B(K_2)$ respectively. Let $K = K_1 \oplus K_2$, $\pi = \pi_1 \oplus \pi_2$, and $V\xi = V_1\xi \oplus V_2\xi$ for $\xi \in H$. Then a straightforward computation shows that if $x \in A$, $\xi, \eta \in H$ then $(V^*\pi(x)V\xi, \eta) = (\varphi(x)\xi, \eta)$, hence φ is decomposable. Q.E.D.

By Lemma 2.1 the definition of decomposable maps makes sense for maps between any C^*-algebras. We shall therefore use the term freely without reference to any underlying Hilbert space.

<u>Lemma 2.2</u>. Let B be a C^*-algebra and M a von Neumann algebra. Suppose φ_α is a decomposable map of B into M for each α in an index set I. Suppose $\varphi(x) = \sum_{\alpha \in I} \varphi_\alpha(x)$ is a bounded operator for all $x \in B$, where the sum converges strongly. Then φ is a decomposable map of B into M.

<u>Proof</u>. Since each $\varphi_\alpha(x) \in M$ so is $\varphi(x)$, since M is strongly closed. By Lemma 2.1 for each $\alpha \in I$ $\varphi_\alpha = \psi_\alpha + \eta_\alpha$ where ψ_α is a completely positive and η_α a completely co-positive map of B into $B(H)$, where M acts on H. Since $0 \leq \psi_\alpha(x) \leq \varphi_\alpha(x)$ for each $x \in B^+$ and similarly for η_α, the sums $\psi = \sum_{\alpha \in I} \psi_\alpha$ and $\eta = \sum_{\alpha \in I} \eta_\alpha$ are well defined positive maps of B into $B(H)$, and $\varphi = \psi + \eta$. Since clearly ψ is completely positive and η completely co-positive, φ is decomposable by Lemma 2.1. Q.E.D.

Lemma 2.3. If B_1, B_2, B_3 are C^*-algebras and φ_i is a decomposable map from B_i into B_{i+1}, $i = 1,2$, then $\varphi_2 \circ \varphi_1$ is a decomposable map from B_1 into B_3.

Proof. We may assume B_i acts on a Hilbert space H_i, $i = 1,2,3$. By Lemma 2.1 $\varphi_i = \psi_i + \eta_i$, where ψ_i is a completely positive, and η_i a completely co-positive map of B_i into $B(H_{i+1})$, $i = 1,2$. Just as a completely positive map from a C^*-algebra into $B(H)$ can be extended to a completely positive map from a larger C^*-algebra into $B(H)$, [1] or [17], so can a completely co-positive map be extended to a completely co-positive map. We can thus extend ψ_2 and η_2 to a completely positive and a completely co-positive map of $B(H_2)$ into $B(H_3)$ respectively. Thus $\psi_2 \circ \psi_1$, $\psi_2 \circ \eta_1$, $\eta_2 \circ \psi_1$, and $\eta_2 \circ \eta_1$ are well defined positive maps. If we can show that $\psi_2 \circ \psi_1$ and $\eta_2 \circ \eta_1$ are completely positive and $\psi_2 \circ \eta_1$ and $\eta_2 \circ \psi_1$ are completely co-positive, then $\varphi_2 \circ \varphi_1 = \psi_2 \circ \psi_1 + \psi_2 \circ \eta_1 + \eta_2 \circ \psi_1 + \eta_2 \circ \eta_1$ is decomposable by Lemma 2.2. For this consider for example $\psi_2 \circ \eta_1$. We have for $n \in \mathbb{N}$,

$$\psi_2 \circ \eta_1 \otimes t_n = (\psi_2 \otimes \iota_n) \circ (\eta_1 \otimes t_n),$$

which is a composition of two positive maps, hence is positive. Thus $\psi_2 \circ \eta_1$ is completely co-positive, and similarly for the other maps. Q.E.D.

Lemma 2.4. Let B be a C^*-algebra and M a von Neumann algebra. Suppose $\{\varphi_\alpha\}_{\alpha \in I}$ is a bounded net of decomposable maps of B into M, and let φ be a point-ultraweak limit point of $\{\varphi_\alpha\}$. Then φ is decomposable.

Proof. Let M act on the Hilbert space H. By Lemma 2.1 for each $\alpha \in I$ there is a completely positive map ψ_α and a completely co-positive map η_α of B into $B(H)$ such that $\varphi_\alpha = \psi_\alpha + \eta_\alpha$. Furthermore $\|\psi_\alpha\| \leq \|\varphi_\alpha\|$, and $\|\eta_\alpha\| \leq \|\varphi_\alpha\|$, so that $\{\psi_\alpha\}_{\alpha\in I}$ and $\{\eta_\alpha\}_{\alpha\in I}$ are bounded nets. Let $\{\varphi_\beta\}$ be a subnet of $\{\varphi_\alpha\}$ which converges to φ. Choose a subnet $\{\psi_\gamma\}$ of $\{\psi_\beta\}$ which converges to a map ψ [11]. Then ψ is completely positive. Choose a subnet $\{\eta_\delta\}$ of $\{\eta_\gamma\}$ which converges to a map η. Then η is completely co-positive. Since subnets of converging nets converge to the same limit, $\varphi = \lim_\delta(\psi_\delta+\eta_\delta) = \lim_\delta \psi_\delta + \lim_\delta \eta_\delta = \psi + \eta$, hence φ is decomposable by Lemma 2.1. Q.E.D.

Lemma 2.5. Let B be a C^*-algebra and H a Hilbert space. Suppose φ is a decomposable map of B into $B(H)$ and that e is a nonzero projection in B. Then the restriction of φ to eBe is decomposable.

Proof. Let $\varphi = V^*\pi V$ be a decomposition for φ as in equation (2.1). Let $W = \pi(e)V$. Then for $x \in B$, $\varphi(exe) = V^*\pi(exe)V = V^*\pi(e)\pi(x)\pi(e)V = W^*\pi(x)W$, so φ restricted to eBe is decomposable. Q.E.D.

Lemma 2.6. Let A and B be C^*-algebras and φ a positive linear map of B into A. Suppose $\pi\circ\varphi$ is decomposable for every representation π in a family of representations of A such that their sum is faithful. Then φ is decomposable.

Proof. Let $\mathcal{F}$ be the given family of representations. Then $\psi = \sum_{\pi\in\mathcal{F}} \oplus\pi$ is a faithful representation of A, and $\psi\circ\varphi = \sum_{\pi\in\mathcal{F}} \oplus\pi\circ\varphi$ is the sum of decomposable maps. By Lemma 2.2 $\psi\circ\varphi$ is decomposable. But then $\varphi = \psi^{-1} \circ (\psi\circ\varphi)$ is decomposable by Lemma 2.3. Q.E.D.

Lemma 2.7. Let B be a unital C^*-algebra and P a unital positive projection of B into itself. Suppose P is decomposable and that $A = P(B_h)$ is a JC-subalgebra of B_h. Then A is a reversible JC-algebra.

Proof. We may assume B acts on Hilbert space H and that P has a decomposition $P = V^*\pi V$ as in (2.1). By assumption A is a JC-algebra, hence $\pi(A)$ is a JC-subalgebra of $B(K)_h$ [7]. Let $a \in A$. Then $V^*\pi(a)^2V = V^*\pi(a^2)V = P(a^2) = a^2 = P(a)^2 = (V^*\pi(a)V)^2$, hence the map $\psi : \pi(A) \to A$ defined by $\psi(\pi(a)) = V^*\pi(a)V$ $(= a)$ is a Jordan homomorphism. In particular, for each $a \in A$, ψ extends to a $*$-homomorphism on the C^*-algebra generated by a, so that $VV^*\pi(a) = \pi(a)VV^*$, i.e. $VV^* \in \pi(A)'$. In particular ψ extends to a $*$-homomorphism on $\pi(A)''$. Let $C = \pi(A)''_h \cap \pi(B)_h$. Then C is the intersection of two reversible JC-algebras, as π is the sum of a homomorphism and an anti-homomorphism [14], hence C is itself a reversible JC-algebra. Since the restriction of ψ to C is the restriction of a $*$-homomorphism on $\pi(A)''$, and $A = \psi(C)$, since by assumption $A = P(B)_h$, A is a reversible JC-algebra. Q.E.D.

We conclude this section with a probably well known result on positive maps.

Lemma 2.8. Let φ be a positive linear map from a C^*-algebra B into another C^*-algebra. Let $N = \{a \in B_h : \varphi(a^2) = 0\}$. Then N is the self-adjoint part of a left ideal in B, hence is in particular the self-adjoint part of a C^*-algebra.

<u>Proof.</u> Let A be a C^*-algebra containing the range of φ. Let $I = \{a \in B : \varphi(a^*a) = 0\}$. Let ρ be a state on A. Then $\rho \circ \varphi$ is a positive linear functional on B, and it is well known that the set $I_\rho = \{a \in B : \rho \circ \varphi(a^*a) = 0\}$ is a uniformly closed left ideal in B. Since $I = \bigcap_\rho I_\rho$, where the intersection is taken over all states of A, the lemma follows.

Q.E.D.

3. Jordan algebras.

We prove some results on JC- and JW-algebras we shall need in the sequel.

Lemma 3.1. Let A be a JC-algebra and z a positive operator in A such that $z \circ a$ is positive for all positive operators $a \in A$. Then z belongs to the center of A.

Proof. It suffices to show that for every irreducible representation π of the C^*-algebra generated by A, $\pi(z)$ is a scalar operator. Since the inequality $z \circ a \geq 0$ continues to hold in the weak closure of $\pi(A)$, we may assume A is an irreducible JW-algebra. If e is a projection in A and $a \in A^+$ then $eae \geq 0$, so that $z \circ eae \geq 0$, and therefore $eze \circ eae = e(z \circ eae)e \geq 0$. Letting e be the sum of two minimal projections in A, cf.[16], we assume A is of type I_2 (if A is of type I_1 the conclusion is trivial). Multiplying z by a scalar we have $z = p + \lambda q$, where $\lambda \geq 0$, and p and q are orthogonal minimal projections in A with sum 1. Let f be a minimal projection in A different from p and q. Then

$$f = \alpha p + \beta v + \bar{\beta} v^* + \gamma q ,$$

where v is a partial isometry from q to p, $\beta v + \bar{\beta} v^* \in A$, and $|\beta|^2 = \alpha\gamma$. Computing we have

$$0 \leq z \circ f = \alpha p + \tfrac{1}{2}\beta(1+\lambda)v + \tfrac{1}{2}\bar{\beta}(1+\lambda)v^* + \lambda\gamma q .$$

Thus we have

$$\lambda\alpha\gamma \geq |\tfrac{1}{2}\beta(1+\lambda)|^2 = \tfrac{1}{4}\alpha\gamma(1+\lambda)^2 .$$

Since $\alpha\gamma \neq 0$ we have $\lambda \geq \frac{1}{4}(1+\lambda)^2$ which implies that $\lambda = 1$.

Thus $z = p + q$ is a scalar operator. Q.E.D.

If A is a reversible JW-factor such that its enveloping von Neumann algebra A'' is not a factor then by [16,Cor.3.5] there are exactly two nonzero orthogonal central projections e and f in A'' with sum 1. Furthermore $eA'' \cong fA''$, and $A''_h = eA$, $fA''_h = fA$, the isomorphism being a Jordan isomorphism.

<u>Lemma 3.2</u>. Let A be a reversible JW-factor such that its enveloping von Neumann algebra A'' is not a factor, and let e and f be the unique nontrivial central projections in A''. Suppose M is a countably decomposable von Neumann algebra containing A such that e and f are properly infinite projections in M. Then there exist three central projections p,q,r in M with sum 1 such that $p \leq e$, $q \leq f$, $pA = pA''_h$, $qA = qA''_h$ and $re \sim rf$ (modulo M).

<u>Proof</u>. If g is a projection in M let c_g denote its central carrier in M. Let $r = c_e c_f$, $p = c_e - r$, $q = c_f - r$. Then $p + q + r = c_e + c_f - r = c_e \vee c_f = 1$. Since $c_f \geq f$, and p is orthogonal to c_f, $p \leq e$, and similarly $q \leq f$. Since $C_{re} = rc_e = c_e c_f = c_{rf}$, and e and f are properly infinite projections in the countably decomposable von Neumann algebra M, $re \sim rf$ in M [5, Ch.III,§8,Cor.5]. Q.E.D.

The next lemma is an extension of [8,Lem.2.5].

<u>Lemma 3.3</u>. Let A be a JC-algebra and N a Jordan ideal in A which is the self-adjoint part of a C^*-algebra. Let B be the C^*-algebra generated by A, and suppose the weak closure $\pi(A)^-$ of $\pi(A)$ is reversible for each $*$-representation π of B annihilating N. Then A is reversible.

Proof. Let $R(A)$ denote the norm closed real $*$-subalgebra of B generated by A. Let $C = R(A)_h$. Then C is a reversible JC-algebra, and we have to show $A = C$, see [14]. Obviously $A \subset C \subset B_h$. If $A \neq C$ there exist two pure states ρ and ω on B such that $\rho|C \neq \omega|C$, while $\rho|A = \omega|A$. Let π_ρ and π_ω be the corresponding GNS-representations on Hilbert spaces H_ρ and H_ω respectively. Then they are either unitarily equivalent or disjoint [6, Prop.5.2.9]. Let $\eta = \frac{1}{2}(\rho+\omega)$, and let π_η be its GNS-representation. We show $\pi_\eta(A)^-$ is reversible.

Suppose first π_ρ and π_ω are unitarily equivalent. Then π_η is unitarily equivalent to $\pi_\rho \otimes 1$ on $H_\rho \otimes \mathbb{C}^2$, in which case $\pi_\eta(A)^- = \pi_\rho(A)^- \otimes \mathbb{C}$ is reversible if $\pi_\rho(A)^-$ is reversible. If $\pi_\rho(N) = 0$, $\pi_\rho(A)^-$ is reversible by hypothesis. Suppose $\pi_\rho(N) \neq 0$. By [7] the C^*-algebra generated by N is a two-sided ideal in B, hence N is itself the self-adjoint part of a two sided ideal J in B. But then $\pi_\rho(J)$ is an irreducible C^*-algebra, so it follows that $\pi_\rho(A)^- \supset \pi_\rho(N)^- = B(H_\rho)_h$, hence $\pi_\rho(A)^-$ is reversible, as is $\pi_\eta(A)^-$.

Next suppose π_ρ and π_ω are disjoint, and again assume $\pi_\eta(N) \neq 0$. Since $\rho|N = \omega|N$, $\rho|J = \omega|J$. Thus $\pi_\rho(J) \neq 0 \neq \pi_\omega(J)$, and as above $\pi_\rho(A)^- = B(H_\rho)_h$, $\pi_\omega(A)^- = B(H_\omega)_h$. By [6, Thm.2.8.3] $\pi_\eta(B)^- = B(H_\rho) \oplus B(H_\omega)$. Let e be the central projection in $\pi_\eta(B)^-$ onto $H_\rho \oplus 0$. Then $\pi_\eta(A)^- e = B(H_\rho)_h$ and $\pi_\eta(A)^-(1-e) = B(H_\omega)_h$. But then from the structure theory for JW-algebras [15] $\pi_\eta(A)^-$ is reversible.

We have thus shown that $\pi_\eta(A)^-$ is reversible in all cases, hence $\pi_\eta(A)^- = \pi_\eta(C)^-$. But if $\bar{\rho}$ and $\bar{\omega}$ are normal states on $\pi_\eta(B)^-$ such that $\bar{\rho} \circ \pi_\eta = \rho$ and $\bar{\omega} \circ \pi_\eta = \omega$, then $\bar{\rho}$ and $\bar{\omega}$ coincide on $\pi_\eta(A)^-$, and therefore on $\pi_\eta(C)^-$. In particular $\rho|C = \omega|C$ contrary to assumption. Therefore $A = C$, and A is reversible. Q.E.D.

4. Preliminaries on projections.

In this section we prove some results on projection maps needed in the sequel. The first result is more general and is due to M. Broise [2], but its proof has never been published.

Lemma 4.1. Let A and B be C^*-algebras and φ a positive linear map of B into A with $\|\varphi\| \leq 1$. Suppose a is a self-adjoint operator in B such that $\varphi(a^2) = \varphi(a)^2$. Then for all $b \in B$ we have

(i) $\varphi(a \circ b) = \varphi(a) \circ \varphi(b)$

(ii) $\varphi(aba) = \varphi(a)\varphi(b)\varphi(a)$.

Proof. We may assume b is self-adjoint. Let $\varepsilon \in \{+1,-1\}$ and $\lambda > 0$. Then by Kadison's Schwarz inequality [10]

$$\varphi((\lambda a + \tfrac{\varepsilon}{\lambda} b)^2) \geq \varphi(\lambda a + \tfrac{\varepsilon}{\lambda} b)^2 ,$$

hence, since $\varphi(a^2) = \varphi(a)^2$,

$$\lambda^{-2}(\varphi(b^2) - \varphi(b)^2) + 2\varepsilon(\varphi(a \circ b) - \varphi(a) \circ \varphi(b)) \geq 0 .$$

Since this holds for both $\varepsilon = +1$ and $\varepsilon = -1$,

$$\lambda^{-2}(\varphi(b^2) - \varphi(b)^2) \geq 2(\varphi(a \circ b) - \varphi(a) \circ \varphi(b)) \geq -\lambda^{-2}(\varphi(b^2) - \varphi(b)^2) .$$

Letting $\lambda \to +\infty$ (i) follows.

To show (ii) note that φ is a *-homomorphism on the abelian C^*-algebra generated by a, as is seen by composing φ with characters on the abelian C^*-algebra generated by $\varphi(a)$. In particular, $\varphi(a^4) = \varphi(a)^4$, hence (i) holds for a^2 as well as a. Thus (ii) follows from the identity $aba = 2a \circ (a \circ b) - a^2 \circ b$. Q.E.D.

Note that the above proof works in the more general situation when B is a JC-algebra.

Lemma 4.2. Let M be a von Neumann algebra, and suppose P is a normal unital positive projection of M into itself. Let $A = P(M_h)$, and suppose A is a JW-subfactor of M_h. Let e be a projection in $M \cap A'$. Then $P(e) = \lambda 1$ with $\lambda \geq 0$. If $\lambda \neq 0$ let $P_e : M_e \to M_e$ be defined by

$$P_e(exe) = \lambda^{-1} P(exe)e.$$

Then P_e is a normal positive projection such that $P_e(e) = e$ and $P_e((M_e)_h) = Ae$.

Proof. If $a \in A^+$ then by Lemma 4.1 $a \circ P(e) = P(a) \circ P(e) = P(a \circ e) \geq 0$, since $a \circ e \geq 0$. By Lemma 3.1 $P(e)$ belongs to the center of A, hence there is $\lambda \geq 0$ such that $P(e) = \lambda 1$. Let P_e be defined as above. Then clearly P_e is normal, positive, and $P_e(e) = e$, and $P_e(eae) = P_e(a \circ e) = \lambda^{-1} P(a \circ e)e = \lambda^{-1}(a \circ P(e))e = ae$, so $P_e((M_e)_h) = Ae$. We finally show P_e is idempotent. For this let $x \in M$. Then from the relation $P(P(x) \circ y) = P(P(x) \circ P(y))$ [8, Lem.2.1] we have

$$\begin{aligned} P_e(P_e(exe)) &= P_e(\lambda^{-1} P(exe)e) \\ &= \lambda^{-2} P(P(exe) \circ e)e \\ &= \lambda^{-2} P(P(exe) \circ P(e))e \\ &= \lambda^{-1} P(P(exe))e \\ &= \lambda^{-1} P(exe)e \\ &= P_e(exe). \end{aligned}$$

Q.E.D.

Recall that if p is a projection in a von Neumann algebra then c_p denotes its central carrier. The next lemma is a variation of the preceding one.

<u>Lemma 4.3.</u> Let M be a von Neumann algebra and P a normal unital positive projection of M into itself. Let $A = P(M_h)$, and suppose A is a JW-subfactor of M_h. Let p be a nonzero projection in M' such that $P(c_p) \neq 0$. Then the map $P_p : Mp \to Mp$ defined by

$$P_p(xp) = P_{c_p}(xc_p)p ,$$

where P_{c_p} is defined in Lemma 4.2, is a normal positive projection such that $P_p(p) = p$, and $P_p((Mp)_h) = Ap$. Furthermore, if $x \in M$ and $P_p(xp) = 0$ whenever p is a countably decomposable projection in M' such that $P(c_p) \neq 0$, then $P(x) = 0$.

<u>Proof.</u> By Lemma 4.2 there is $\lambda > 0$ such that $P(c_p) = \lambda 1$. Thus we have by Lemma 4.2

$$P_p(xp) = \lambda^{-1}P(xc_p)c_p p = \lambda^{-1}P(xc_p)p .$$

Hence it is straightforward to show that P_p is a normal positive projection. Since by Lemma 4.2 $P_{c_p}((Mc_p)_h) = Ac_p$ we have $P_p((M_p)_h) = Ap$. Finally, suppose $P_p(xp) = 0$ whenever p is a countably decomposable projection in M' such that $P(c_p) \neq 0$. Then by definition of P_p and the isomorphism $Mc_p \to Mp$, $P(xc_p)c_p = 0$. But if $\lambda = P(c_p) \neq 0$ we have $0 = P(P(xc_p)c_p) = \lambda P(P(xc_p)) = \lambda P(xc_p)$, so $P(xc_p) = 0$. Furthermore, if $P(c_p) = 0$ then by Lemma 2.8 $P(xc_p) = 0$. We have thus shown that $P(xc_p) = 0$ for all countably decomposable projections p in M'. Since every projection in M' majorizes a countably decomposable projection, an easy Zorn's Lemma argument yields a family $\{p_\alpha\}$ of countably decomposable projections in M' with $\Sigma c_{p_\alpha} = 1$. Since P is normal, P is ultraweakly continuous, hence $P(x) = \sum_\alpha P(xc_{p_\alpha}) = 0$. Q.E.D.

Lemma 4.4. Let M be a von Neumann algebra and P a normal unital positive projection of M into itself. Then there exist a von Neumann algebra N containing M such that every nonzero projection in M is infinite in N, and a normal unital positive projection Q of N onto $P(M)$ extending P. Furthermore, Q is decomposable if and only if P is decomposable.

Proof. Let M act on a Hilbert space H, let K be a separable infinite dimensional Hilbert space, and let ρ be a vector state on $B(K)$. Then the projection $\iota\otimes\rho$ is a normal unital positive projection of $M\otimes B(K)$ onto $M\otimes\mathbb{C}$. Hence the projection

$$P\otimes\rho : M\otimes B(K)\to P(M)\otimes\mathbb{C} ,$$

which is the composition of the maps $\iota\otimes\rho$ and $P\otimes\iota_{\mathbb{C}}$, is positive and normal, where ι_N denotes the identity map of a von Neumann algebra N on itself. If we identify M with $M\otimes\mathbb{C}$ and $P(M)$ with $P(M)\otimes\mathbb{C}$ we let $Q = P\otimes\rho$ and $N = M\otimes B(K)$. Thus the existence of N and Q is proved. Finally, if P is decomposable, so is $P\otimes\iota_{\mathbb{C}}$, hence $Q = (P\otimes\iota_{\mathbb{C}})\circ(\iota_M\otimes\rho)$ is decomposable. Conversely, if Q is decomposable then so is its restriction to M, hence P is decomposable. Q.E.D.

Lemma 4.5. Let B be a unital C^*-algebra and P a unital positive projection of B into itself. Suppose $A = P(B_h)$ is a JC-subalgebra of B_h. Let P^{**} be the extension of P to the second dual B^{**} of B. Then P^{**} is a normal unital positive projection of $(B^{**})_h$ onto A^{**}. Suppose further that for each minimal central projection p in A^{**} such that $A^{**}p$ is of type I, the projection P_p^{**} of $(B^{**})_p$ into itself defined by $P_p^{**}(pxp) = P^{**}(pxp)$ $(= P^{**}(x)p)$ is decomposable. Then P is decomposable.

Proof. It is clear that P^{**} and P_p^{**} are normal positive projections with self-adjoint images A^{**} and $A^{**}p$ respectively (note that by Lemma 4.1 $P^{**}(pxp) = pP^{**}(x)p = P^{**}(x)p$).

Let $\mathfrak{F}$ be the set of minimal central projections p in A^{**} such that $A^{**}p$ is a JW-factor of type I, and let π_p be the $*$-representation of the C^*-algebra (A) generated by A defined by $\pi_p(a) = ap$, where we consider B as a subalgebra of B^{**}. We then have $\pi_p(A)^- = A^{**}p$ is a JW-factor of type I, and $\pi_p(P(x)) = P(x)p = P^{**}(x)p = P_p^{**}(pxp)$, which is the composition of the two decomposable maps $x \to pxp$ and P_p^{**}, hence is decomposable by Lemma 2.3. In order to complete the proof of the lemma it suffices by Lemma 2.6 to show the family $\{\pi_p : p \in \mathfrak{F}\}$ is faithful on (A). For this let $a \in (A)$ and $ap = 0$ for all $p \in \mathfrak{F}$. Since the factor representations of (A) of type I form a faithful family it suffices to show that whenever q is a minimal central projection in $(A)^{**}$ such that $(A)^{**}q$ is of type I then $aq = 0$. Given such a projection q let e be its central carrier in A^{**}. Then e is a minimal central projection in A^{**}. Indeed if f is a central projection in A^{**} and $f \leq e$ then f is central in $(A)^{**}$, so either $fq = q$ or $fq = 0$. In the former case $e = f$ and in the latter $e-f \geq q$, so $f = 0$. Thus e is minimal, and $A^{**}e$ is a JW-factor. Since $A^{**}q$ generates $(A)^{**}q$, $A^{**}q$ is of type I by [16, Thm.4.1], hence $A^{**}e$, being isomorphic to $A^{**}q$, is of type I. But then by hypothesis $ae = 0$, and therefore $aq = 0$. Q.E.D.

5. Projections onto reversible JW-factors of type I_2.

There are up to isomorphisms of the enveloping C^*-algebras four different reversible JW-factors of type I_2, see [9] or [15], namely:

(i) S_2 - the real symmetric matrices in M_2.

(ii) $(M_2)_h$ - the self-adjoint matrices in M_2.

(iii) Q_2 - the self-adjoint matrices in M_4 of the form

$$x = \begin{pmatrix} \alpha 1 & q \\ q^* & \beta 1 \end{pmatrix},$$

where 1 is the identity in M_2, $\alpha,\beta \in \mathbb{R}$, and q is a quaternion represented as a 2×2 matrix.

(iv) D_2 - the self-adjoint matrices in M_4 of the form

$$x = \begin{pmatrix} a & 0 \\ 0 & a^t \end{pmatrix},$$

where a is a self-adjoint matrix in M_2, and a^t is the transpose of a.

We shall in the present section classify all unital positive projections of M_n onto the above reversible JW-factors, where $n = 2$ in cases (i) and (ii), and $n = 4$ in cases (iii) and (iv). There is trivially only one projection onto $(M_2)_h$, namely the identity map.

Lemma 5.1. Let P be a positive projection of M_4 into itself such that $P((M_4)_h) = Q_2$. Then if $a,b \in M_2$ we have

$$P\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} \tau(a)1 & 0 \\ 0 & \tau(b)1 \end{pmatrix},$$

where τ is the normalized trace, and 1 is the identity in M_2.

<u>Proof</u>. We have $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \in Q_2$ and $\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \in Q_2$, hence if $x \in M_2$ then

$$P\left(\begin{pmatrix}x & 0\\ 0 & 0\end{pmatrix}\right) = \begin{pmatrix}\rho(x)1 & 0\\ 0 & 0\end{pmatrix}, \quad P\left(\begin{pmatrix}0 & 0\\ 0 & x\end{pmatrix}\right) = \begin{pmatrix}0 & 0\\ 0 & \omega(x)1\end{pmatrix},$$

where ρ and ω are states on M_2. We show $\rho = \omega = \tau$. Let $s \in M_2$ be a quaternion of norm 1. Then $s^*s = ss^* = 1$, and $S = \begin{pmatrix}0 & s^*\\ s & 0\end{pmatrix} \in Q_2$. Let $x = \begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}$ with $a,b \in (M_2)_h$. Then we have, using Lemma 4.1,

$$\begin{aligned}\begin{pmatrix}\rho(s^*bs) & 0\\ 0 & \omega(sas^*)\end{pmatrix} &= P\left(\begin{pmatrix}s^*bs & 0\\ 0 & sas^*\end{pmatrix}\right)\\ &= P(SxS)\\ &= SP(x)S\\ &= \begin{pmatrix}\omega(b) & 0\\ 0 & \rho(a)\end{pmatrix}.\end{aligned}$$

Since this in particular holds for $s = 1$ we have $\rho = \omega$. Thus $\rho(a) = \rho(sas^*)$ for all s among the quaternions Q in M_2. In particular Q is contained in the centralizer M_ρ of ρ in M_2. But then the C^*-algebra generated by Q is contained in M_ρ, hence $M_\rho = M_2$, since Q is irreducible in M_2. Thus $\rho = \tau$.

Q.E.D.

An easy modification of the argument in the beginning of the above proof yields the following result for S_2.

<u>Lemma 5.2</u>. Let P be a positive projection of M_2 into itself such that $P((M_2)_h) = S_2$. Then

$$P\left(\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\right) = \begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}, \quad \text{for all } a,b \in \mathbb{C}.$$

Lemma 5.3. Let P be a positive projection of M_4 (resp. M_2) into itself such that $P((M_4)_h) = Q_2$ (resp. $P((M_2)_h) = S_2$). Then P equals the unique positive projection with the property that $\mathrm{Tr}(xa) = \mathrm{Tr}(P(x)a)$ for all $a \in Q_2$ (resp. $a \in S_2$) and all $x \in M_4$ (resp. $x \in M_2$), where Tr is the usual trace on M_4 (resp. M_2).

Proof. Let M denote M_4 when we consider Q_2 and M_2 when we consider S_2, and let A denote either Q_2 or S_2. Then A is a JW-factor of type I_2, hence is a spin factor [15, Thm.7.1]. Thus by [8 ,Lem.2.3] there exists a positive projection P_1 of M into itself with $P_1(M_h) = A$, defined by the equation $\mathrm{Tr}(xa) = \mathrm{Tr}(P_1(x)a)$ for all $a \in A$, $x \in M_h$. By Lemma 5.1 and 5.2 $\mathrm{Tr}(P(x)) = \mathrm{Tr}(x)$ for all $x \in M$. Thus if $x \in M_h$, $a \in A$ we have by Lemma 4.1

$$\begin{aligned}\mathrm{Tr}(P(x)a) = \mathrm{Tr}(P(x)\circ a) = \mathrm{Tr}(P(x\circ a)) &= \mathrm{Tr}(x\circ a) \\ &= \mathrm{Tr}(xa)\,,\end{aligned}$$

hence $P = P_1$. Q.E.D.

Lemma 5.4. Let by Lemma 5.3 P_2 denote the unique positive projection of $(M_2)_h$ onto S_2. Let N be a von Neumann algebra and P a normal positive projection of the von Neumann algebra $M_2 \otimes N$ into itself such that $P((M_2 \otimes N)_h) = S_2 \otimes \mathbb{R}1$. Then there exists a normal state ρ on N such that $P = P_2 \otimes \rho$.

Proof. As pointed out in the proof of Lemma 4.2 it follows from Lemma 3.1 that P restricted to $\mathbb{C}1 \otimes N$ is a state ρ. If $b \in (\mathbb{C}1 \otimes N)^+$ and $P(b) = 1$ then for $a \in S_2 \otimes \mathbb{R}1$ we have by [8 ,Lem.2.1] that $P(ab) = P(a\circ b) = P(a\circ P(b)) = a\rho(b) = a$. Thus $P_b(x) = P(x\circ b)$ is a positive projection of $(M_2 \otimes \mathbb{C}1)_h$ onto $S_2 \otimes \mathbb{R}1$, so by Lemma 5.3 $P(x\circ b) = P_2(x)\rho(b)$ for $x \in M_2$. Since each $c \in \mathbb{C}1 \otimes N$ is a linear combination of b's as above, $P(x \circ c) = P_2(x)\rho(c)$ for all $x \in M_2 \otimes \mathbb{C}1$ and $c \in \mathbb{C}1 \otimes N$, i.e. $P = P_2 \otimes \rho$.

Q.E.D.

We are now in position to study projections of $(M_4)_h$ onto D_2 (case iv).

Lemma 5.5. Let D_2 be represented as matrices $\begin{pmatrix} a & 0 \\ 0 & a^t \end{pmatrix}$ with $a \in (M_2)_h$, and let $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, where 1 is the identity in M_2. Let P be a positive projection of $(M_4)_h$ onto D_2. Then

$$P(exf + fxe) = 0 \quad \text{for all} \quad x \in M_4.$$

Proof. Let e_{ij}, $i,j = 1,2$, denote the usual matrix units in M_2, and identify M_4 with $M_2 \otimes M_2$ in the usual way, so in particular $e = 1 \otimes e_{11}$, $f = 1 \otimes e_{22}$ and $D_2 = \{a \otimes e_{11} + a^t \otimes e_{22} : a \in (M_2)_h\}$. We shall compute several values for P. The first is

(5.1) $$P(e_{11} \otimes (e_{12}+e_{21})) = z_{11}(e_{11} \otimes 1), \quad z_{11} \in [-1,1].$$

We have $(e_{22} \otimes 1) \circ (e_{11} \otimes (e_{12}+e_{21})) = 0$, hence by Lemma 4.1

$$0 = P((e_{22} \otimes 1) \circ (e_{11} \otimes (e_{12}+e_{21}))) = (e_{22} \otimes 1) \circ P(e_{11} \otimes (e_{12}+e_{21})).$$

Let $P(e_{11} \otimes (e_{12}+e_{21})) = a \otimes e_{11} + a^t \otimes e_{22}$ with $a = (a_{ij}) \in (M_2)_h$. Then $e_{22} \circ a = 0$, hence $a_{12} = a_{21} = a_{22} = 0$, and (5.1) follows.

The same argument shows

(5.2) $$P(e_{22} \otimes (e_{12}+e_{21})) = z_{22}(e_{22} \otimes 1), \quad z_{22} \in [-1,1].$$

We next show

(5.3) If $w \in \mathbb{C}$ there is $z \in \mathbb{C}$ such that

$$P(we_{12} \otimes e_{12} + \bar{w}e_{21} \otimes e_{21}) = z(e_{12} \otimes e_{11} + e_{21} \otimes e_{22}) + \bar{z}(e_{21} \otimes e_{11} + e_{12} \otimes e_{22}).$$

Indeed, if $a = e_{11} - e_{22} \in M_2$ then

$$(a\otimes 1)\circ(we_{12}\otimes e_{12} + \bar{w}e_{21}\otimes e_{21}) = 0.$$

Thus if $b = (b_{ij}) \in (M_2)_h$ satisfies

$$P(we_{12}\otimes e_{12} + \bar{w}e_{21}\otimes e_{21}) = b\otimes e_{11} + b^t \otimes e_{22}$$

then by Lemma 4.1 we have $0 = a\circ b$, hence $b_{11} = b_{22} = 0$, and (5.3) follows with $z = b_{12}$.

An analogous computation yields

(5.4) If $w \in \mathbb{C}$ there is $z \in \mathbb{C}$ such that

$$P(we_{21}\otimes e_{12} + \bar{w}e_{12}\otimes e_{21}) = z\,(e_{12}\otimes e_{11} + e_{21}\otimes e_{22}) + \bar{z}\,(e_{21}\otimes e_{11} + e_{12}\otimes e_{22}).$$

We next show that if z_{11} and z_{22} are as in (5.1) and (5.2) then

(5.5) $z_{11} = -z_{22}$.

Indeed let $a = ie_{12} - ie_{21} \in M_2$. Then

$$(a\otimes e_{11} + a^t\otimes e_{22})\circ(1\otimes(e_{12}+e_{21})) = 0.$$

Let $b = (b_{ij}) \in (M_2)_h$ be defined by

$$P(1\otimes(e_{12}+e_{21})) = b\otimes e_{11} + b^t\otimes e_{22}.$$

Then by Lemma 4.1

$$0 = (a\otimes e_{11} + a^t\otimes e_{22})\circ(b\otimes e_{11} + b^t\otimes e_{22})$$

$$= (a\circ b)\otimes e_{11} + (a\circ b)^t\otimes e_{22}.$$

It follows that $b_{11} = -b_{22}$ and $b_{21} = b_{12}$. However, by (5.1) and (5.2) we have

$$P(1\otimes(e_{12}+e_{21})) = P(e_{11}\otimes(e_{12}+e_{21})) + P(e_{22}\otimes(e_{12}+e_{21}))$$
$$= z_{11}(e_{11}\otimes 1) + z_{22}(e_{22}\otimes 1).$$

Consequently $b_{12} = b_{21} = 0$, and $z_{11} = b_{11} = -b_{22} = -z_{22}$, so (5.5) follows.

Let R be the projection of D_2 onto $S_2\otimes\mathbb{R}1$ defined by

$$R(a\otimes e_{11} + a^t\otimes e_{22}) = \tfrac{1}{2}(a+a^t)\otimes 1.$$

Then $R\circ P$ is a projection of M_4 into itself such that $R\circ P((M_4)_h) = S_2\otimes\mathbb{R}1$. By Lemma 5.4 there exists a state ρ on M_2 such that if P_2 is the projection of $(M_2)_h$ onto S_2 defined by $P_2(a) = \frac{1}{2}(a+a^t)$, then $R\circ P = P_2\otimes\rho$. Thus we have

$$R\circ P(e_{11}\otimes(e_{12}\otimes e_{21})) = P_2\otimes\rho(e_{11}\otimes(e_{12}+e_{21}))$$
$$= P_2(e_{11})\rho(e_{12}+e_{21})$$
$$= \rho(e_{12}+e_{21})e_{11}\otimes 1.$$

Comparing this with (5.1) we have

$$z_{11} = \rho(e_{12}+e_{21}).$$

Similarly we have

$$z_{22} = \rho(e_{12}+e_{21}).$$

But by (5.5) $z_{11} = -z_{22}$, hence we have shown

(5.6) $$z_{11} = z_{22} = \rho(e_{12}+e_{21}) = 0.$$

The last equation can be improved as follows. If $z \in \mathbb{C}$ then we have

$$(5.7) \qquad P(ze_{11}\otimes e_{12} + \bar{z}e_{11}\otimes e_{21}) = 0.$$

Indeed, let $w \in \mathbb{C}$, $w^2 = z$, and let $a = w\, e_{12} + \bar{w}\, e_{21}$, and $\tilde{a} = a \otimes e_{11} + a^t \otimes e_{22}$. Then a straightforward computation yields

$$z\, e_{11} \otimes e_{12} + \bar{z}\, e_{11} \otimes e_{21} = \tilde{a}(e_{22} \otimes (e_{12}+e_{21}))\tilde{a}\,.$$

Thus by Lemma 4.1 and (5.6) we have

$$P(ze_{11}\otimes e_{12} + \bar{z}e_{11}\otimes e_{21}) = P(\tilde{a}(e_{22} \otimes (e_{12}+e_{21}))\tilde{a}) = 0\ ,$$

and (5.7) follows. Similarly we have for $z \in \mathbb{C}$,

$$(5.8) \qquad P(ze_{22}\otimes e_{12} + \bar{z}e_{22}\otimes e_{21}) = 0.$$

Combining (5.7) and (5.8) with (5.3) and (5.4) we have shown that if $a \in M_2$ then there is $z \in \mathbb{C}$ such that

$$(5.9)\ P(a\otimes e_{12} + a^*\otimes e_{21}) = z(e_{12}\otimes e_{11} + e_{21}\otimes e_{22}) + \bar{z}(e_{21}\otimes e_{11} + e_{12}\otimes e_{22}).$$

Since $e \in (D_2)'$, by Lemma 3.1 there is $\lambda \in [0,1]$ such that $P(e) = \lambda 1$, $P(f) = (1-\lambda)1$. If $\lambda = 0$ or 1, then by Lemma 2.8 $P(exf + fxe) = 0$ for all $x \in M_4$. Therefore assume $0 < \lambda < 1$, and let

$$h = \tfrac{1}{2}\lambda^{-1}e + \tfrac{1}{2}(1-\lambda)^{-1}f.$$

Then $P(h) = 1$ and $h \in (D_2)'$. Furthermore, a straightforward computation shows that if $x = \Sigma\, a_{ij} \otimes e_{ij} \in M_4$, $a_{ij} \in M_2$, then

$$h^{\frac{1}{2}} x\, h^{\frac{1}{2}} = \tfrac{1}{2}\lambda^{-1}a_{11}\otimes e_{11} + \tfrac{1}{2}(\lambda(1-\lambda))^{-\frac{1}{2}}(a_{12}\otimes e_{12} + a_{21}\otimes e_{21}) + \tfrac{1}{2}(1-\lambda)^{-1}a_{22}\otimes e_{22}.$$

Hence by (5.9), if $a \in M_2$ then there is $z \in \mathbb{C}$ such that

(5.10) $P(h^{\frac{1}{2}}(a\otimes e_{12} + a^*\otimes e_{21})h^{\frac{1}{2}}) = z(e_{12}\otimes e_{11} + e_{21}\otimes e_{22}) + \bar{z}(e_{21}\otimes e_{11} + e_{12}\otimes e_{22})$.

Let $P_h(x) = P(h^{\frac{1}{2}}xh^{\frac{1}{2}})$, $x \in M_4$. Then a similar computation as in the proof of Lemma 4.2 shows that P_h is a positive projection of $(M_4)_h$ onto D_2 such that $P_h(e) = P_h(f) = \frac{1}{2}1$. But then if $a \in (M_2)_h$, by Lemma 4.1

$$\begin{aligned} P_h(a\otimes e_{11}) &= P_h((a\otimes e_{11} + a^t\otimes e_{22})\circ e) \\ &= (a\otimes e_{11} + a^t\otimes e_{22})\circ P_h(e) \\ &= \tfrac{1}{2}(a\otimes e_{11} + a^t\otimes e_{22}), \end{aligned}$$

and similarly for $a\otimes e_{22}$. Thus by (5.10) we have $\mathrm{Tr}(P_h(x)) = \mathrm{Tr}(x)$ for all $x \in (M_4)_h$. Since D_2 is a JW-factor of type I_2, it is a spin factor [15], hence it follows as in Lemma 5.3 that P_h is the unique positive projection Q of $(M_4)_h$ onto D_2 such that $\mathrm{Tr}(Q(x)\circ a) = \mathrm{Tr}(x\circ a)$ for all $a \in D_2$. Since the projection $x \to P_h(exe + fxf)$ has this property, we have shown

$$P_h(x) = P_h(exe + fxf),$$

hence $P_h(exf + fxe) = 0$ for all $x \in M_4$. But then if $x \in M_4$

$$\begin{aligned} P(exf + fxe) &= P_h(h^{-\frac{1}{2}}(exf + fxe)h^{-\frac{1}{2}}) \\ &= P_h(e(h^{-\frac{1}{2}}xh^{-\frac{1}{2}})f + f(h^{-\frac{1}{2}}xh^{-\frac{1}{2}})e) \\ &= 0 . \end{aligned}$$

Q.E.D.

It is easy from the last lemma to classify all positive projections of $(M_4)_h$ onto D_2. However, such a classification is immediate from Proposition 6.4 below.

6. Projections onto reversible JW-factors of type I.

In this section we shall classify all normal unital positive projections P of a von Neumann algebra M into itself such that $P(M_h)$ is a reversible JW-factor of type I.

Lemma 6.1. Let A be an irreducible reversible JW-algebra acting on a Hilbert space H. Then there exists a unique normal positive projection of $B(H)_h$ onto A.

Proof. By [8, Thm.2.2] there exists a normal positive projection P of $B(H)_h$ onto A. In order to show uniqueness let R be another normal positive projection of $B(H)_h$ onto A. By [16] A is of type I. If e and f are mutually orthogonal minimal projections in A, and $g = e+f$ it clearly suffices to show the restrictions $P|B(H)_g = R|B(H)_g$. But this is immediate from Lemma 5.3 and the classification of irreducible JW-factors in [15]. Q.E.D.

The next result classifies all normal unital projections of a von Neumann algebra onto a reversible JW-factor of type I, whose enveloping von Neumann algebra is also a factor. The proof is a trivial modification of the proof of Lemma 5.4 using Lemma 6.1 rather than Lemma 5.3, and is therefore omitted.

Proposition 6.2. Let N be a von Neumann algebra and M a type I factor. Let A be a reversible JW-factor such that $M = A''$. Suppose P is a normal positive projection of $M \otimes N$ into itself such that $P((M\otimes N)_h) = A \otimes \mathbb{R}1$. Then $P = Q \otimes \rho$, where Q is the unique positive projection of M_h onto A found in Lemma 6.1, and ρ is a normal state on N.

We next consider the case when A is a reversible JW-factor of type I such that its enveloping von Neumann algebra A'' is not a factor.

Lemma 6.3. Let A be a reversible JW-factor of type I such that its enveloping von Neumann algebra A'' is not a factor. Let e and f be the unique nonzero minimal projections in the center of A''. Suppose M is a von Neumann algebra containing A such that every nonzero projection in A is infinite in M and that P is a normal positive projection of M into itself such that $P(M_h) = A$. Then

(6.1) $$P(exf + fxe) = 0 \quad \text{for all} \quad x \in M.$$

Proof. By Lemma 4.3 we may assume M is countably decomposable. By Lemma 3.2 there are three mutually orthogonal central projections p, q and r in M with sum 1 such that $p \leq e$, $q \leq f$, $pA = pA''_h$, $qA = qA''_h$, and $re \sim rf$ (modulo M). Note that since $pf = 0 = qe$, and p and q are central projections in M, we have for $x \in M$

(6.2) $$P(exf + fxe) = P((re)x(rf) + (rf)x(re)).$$

Let $\lambda 1 = P(r)$, cf. Lemma 4.2. If $\lambda = 0$ then by Lemma 2.8 (6.1) is immediate from (6.2). Suppose $\lambda \neq 0$ and let P_r be the projection of Mr into itself defined by $P_r(rx) = \lambda^{-1}P(rx)r, x \in M$, cf. Lemma 4.2. Since $r \in A'$ and $re \neq 0 \neq rf$, and e and f are the unique minimal central projections in A'' r
r is separating for A'', hence for $''A$. Thus $P(rx) = 0$ if and only if $P(rx)r = 0$ for $x \in M$. Hence $P(rx) = 0$ if and only if $P_r(rx) = 0$. In particular we have shown that (6.1) holds if and only if

$$P_r((er)(xr)(fr) + (fr)(xr)(er)) = 0, \quad x \in M.$$

We may therefore assume $r = 1$, and $e \sim f$ in M. By [16, Cor.3.5] the map $\chi : A''e \to A''f$ by $\chi : ae \to a \to af$ for $a \in A$, is a Jordan isomorphism. In particular $A''e$ and $A''f$ are isomorphic type I factors. We note for later references that χ is a *-anti-isomorphism. Now, χ is either an isomorphism or an anti-isomorphism. If it were an isomorphism, A would be the self-adjoint part of the von Neumann algebra $\{ae + \chi(a)f : a \in A''e\}$, a case we have excluded. Thus χ is a *-anti-isomorphism.

Let $(e_\alpha)_{\alpha \in I}$ be an orthogonal family of minimal projections in Ae with sum e, and let $(f_\alpha)_{\alpha \in I}$ be the corresponding family in Af obtained by χ, so $f_\alpha = \chi(e_\alpha)$. Let $o \in I$, and let v_α and w_α be partial isometries in Ae and Af respectively such that $v_\alpha^* v_\alpha = e_o$, $v_\alpha v_\alpha^* = e_\alpha$, $w_\alpha^* w_\alpha = f_o$, $w_\alpha w_\alpha^* = f_\alpha$ and $v_o = e_o$, $w_o = f_o$. Since the central carrier c_{e_o} of e_o is the same as that of e, which is 1, $c_{e_o} = 1$. Similarly $c_{f_o} = 1$. By assumption e_o and f_o are infinite projections in M, and M is countably decomposable. Thus by [5, Ch.III, § 8, Cor.5] $e_o \sim f_o$ in M. Let s be a partial isometry in M such that $s^*s = e_o$, $ss^* = f_o$. Let $u_\alpha = w_\alpha s v_\alpha^*$. Then a straightforward computation shows $u_\alpha^* u_\alpha = e_\alpha$, $u_\alpha u_\alpha^* = f_\alpha$. Let $u = \sum_{\alpha \in I} u_\alpha$, where the convergence is in the strong topology. Since $u_\alpha^* u_\beta = 0$ for $\alpha \neq \beta$, $u^* u = e$, $uu^* = f$. Furthermore $u v_\alpha u^* = u_\alpha v_\alpha u_o^* = w_\alpha$, so that u defines a *-isomorphism of $A''e$ onto $A''f$ by $x \to uxu^*$.

Let R be the von Neumann algebra generated by A and u. Since $eu = 0 = u^*e$, and e is central in A'', it follows from formulas like $au = 0$ if $a \in Ae$, $au = u(u^*au) \in uAe$, that $R_e = A''e$, $R_f = A''f$. In particular R is a type I factor, and P

restricted to R_h is a normal positive projection onto A. We show that (6.1) holds for all $x \in R$. Since P is normal, P is ultraweakly continuous, hence

$$P(exf + fxe) = \sum_{\alpha,\beta \in I} P(e_\alpha x f_\beta + f_\beta x e_\alpha).$$

It therefore suffices to show

$$(6.3) \qquad P(e_\beta x f_\beta + f_\beta x e_\alpha) = 0 \quad \alpha, \beta \in I, \quad x \in R.$$

Let $g_\alpha = e_\alpha + f_\alpha$ for $\alpha \in I$. Since $f_\alpha = \chi(e_\alpha)$, $g_\alpha \in A$. Note that if we show

$$(6.4) \qquad P(g_\alpha x g_\beta + g_\beta x g_\alpha) = 0$$

whenever $\alpha \neq \beta$, (6.3) follows in particular when $\alpha = \beta$ by choosing an index different from α, and using (6.4). Therefore assume $\alpha \neq \beta$, and let $g = g_\alpha + g_\beta$. Then R_g is isomorphic to M_4, and A_g is isomorphic to D_2. Hence (6.4) follows from Lemma 5.5, and (6.3) follows. Thus (6.1) holds for all $x \in R$. Since the restriction $P|R$ is an arbitrary normal positive projection of R_h onto A, (6.1) holds for any normal positive projection of R_h onto A and $x \in R$.

Let N be the relative commutant of R in M. Since R is a type I subfactor of M operators of the form $\sum_{i=1}^{n} x_i b_i$ with $x_i \in R$, $b_i \in N$ are ultraweakly dense in M. If $b \in N^+$, $P(b)$ is a scalar operator by Lemma 4.2. Thus if $P(b) = 1$ it follows as in the proof of Lemma 5.4 that the map $P_b(x) = P(xb)$ is a normal positive projection of R_h onto A. Thus by the preceding paragraph

$$P(xb) = P_b(x) = P_b(exe + fxf) = P((exe + fxf)b)$$

$$= P(e(xb)e + f(xb)f).$$

Since operators of the form $\Sigma x_i b_i$ with $b_i \in N^+$ are ultraweakly dense in M and P is ultraweakly continuous, $P(x) = P(exe + fxf)$ for all $x \in M$. In particular (6.1) follows. Q.E.D.

<u>Proposition 6.4.</u> Suppose M is a von Neumann algebra and A is a reversible JW-subfactor of M of type I such that A'' is not a factor and every nonzero projection in A is infinite in M. Suppose P is a normal positive projection of M_h onto A, and let e and f be the unique minimal central projections in A''. Let R_e and R_f be the projections of A''_h onto A defined by

$$R_e(x) = R_e(xe) = \bar{x}, \quad \text{where} \quad \bar{x} \in A \quad \text{satisfies} \quad \bar{x}e = xe.$$

$$R_f(x) = R_f(xf) = \tilde{x}, \quad \text{where} \quad \tilde{x} \in A \quad \text{satisfies} \quad \tilde{x}f = xf.$$

Let $\lambda \in [0,1]$ be defined by $\lambda 1 = P(e)$, cf. Lemma 4.2. Then if $\lambda = 0$ (resp. $\lambda = 1$) there is a normal positive projection Q of M_f onto $A''f$ (resp. M_e onto $A''e$) such that for $x \in M$,

$$P(x) = R_f \circ Q(fxf) \qquad (\text{resp. } P(x) = R_e \circ Q(exe)).$$

If $0 < \lambda < 1$ there exists a normal positive projection Q of M onto A'' such that

$$P(x) = (\lambda R_e + (1-\lambda) R_f) \circ Q(x) \ , \qquad x \in M.$$

<u>Proof.</u> Assume first $0 < \lambda < 1$. Let P_e be the projection of M_e onto $A''e$ defined in Lemma 4.2 by $P_e(exe) = \lambda^{-1} P(exe)e$, and similarly define P_f. Let

$$Q(x) = P_e(exe) + P_f(fxf) .$$

Then Q is normal positive projection of M onto A''. If $x \in M$, then by Lemma 6.3

$$\begin{aligned} P(Q(x)) &= P(\lambda^{-1}P(exe)e + (1-\lambda)^{-1}P(fxf)f) \\ &= P(exe) + P(fxf) \\ &= P(x) , \end{aligned}$$

so that $P = (P|A'')\circ Q$. Finally if $x \in A_h''$ let x_1 and x_2 be the unique operators in A such that $x_1 e = xe$, $x_2 f = xf$. Then we have by Lemma 4.1,

$$\begin{aligned} P(x) &= P(xe) + P(xf) \\ &= P(x_1 e) + P(x_2 f) \\ &= x_1 P(e) + x_2 P(f) \\ &= \lambda x_1 + (1-\lambda) x_2 \\ &= \lambda R_e(x) + (1-\lambda) R_f(x). \end{aligned}$$

Thus the proposition follows for $\lambda \in (0,1)$. If for example $\lambda = 1$ an inspection of the above argument shows that $P(x) = R_e \circ P_e(exe)$, so $Q(x) = P_e(exe)$ satisfies the requirements. Q.E.D.

Remark 6.5. Note that if R is a type I subfactor of a von Neumann algebra M, M can be identified with $R \otimes N$, where N is the relative commutant of R in M. Thus a normal positive projection of M onto R is by Proposition 6.2 of the form $P = \iota \otimes \rho$, where ι is the identity map on R, and ρ is a normal state on N. This remark gives a complete description of the expectation Q in Proposition 6.4. In particular, it is immediate that Q is completely positive, a fact which also follows from the more general result of Tomiyama [18]

7. The main results.

Let B be a unital C^*-algebra and P a unital positive projection of B into itself. Let $A = P(B_h)$. Let P^{**} denote the normal extension of P to the second dual B^{**} of B, and consider B as a C^*-subalgebra of B^{**}. Let r be the support projection of P^{**} in B^{**}. By [8, Lem.1.2] $r \in A'$, and the map $A \to Ar$ is a Jordan isomorphism of A with the Jordan product $a * b = P(a \circ b)$ onto the JC-algebra Ar. We say P is <u>weakly decomposable</u> if the inverse map $ra \to a$ of Ar onto A has a decomposable extension to the C^*-subalgebra of B^{**} generated by Ar.

<u>Theorem 7.1</u>. Let B be a unital C^*-algebra and P a unital positive projection of B into itself. Let $A = P(B_h)$ and $N = \{a \in B_h : P(a^2) = 0\}$. Then P is decomposable if and only if P is weakly decomposable and $A + N$ is a reversible JC-subalgebra of B_h.

<u>Proof</u>. Assume P is weakly decomposable and that $A + N$ is a reversible JC-subalgebra of B. As above let P^{**} be the extension of P to B^{**} and r its support. Since $r \in A'$ and $rN = 0$, $r \in (A+N)'$, hence Ar is a reversible JC-algebra. Since the map $x \to rxr$ on B^{**} is decomposable, and the map $Ar \to A$ by $ar \to a$ is decomposable by assumption, we know by Lemma 2.3 that P is decomposable if the projection $P_r^{**} : (B_r^{**})_h \to A^-r$ defined by $P_r^{**}(rxr) = P^{**}(rxr)r$, is decomposable. We have thus reduced the proof to the case when P is a unital positive projection of the C^*-algebra B into itself such that $P(B_h) = A$ is a reversible JC-subalgebra of B_h. By Lemma 4.5 in order to show P is decomposable we may assume B is a von Neumann algebra, A is

a reversible JW-subfactor of type I of B_h, and P is a normal positive projection. Furthermore, by Lemma 4.4 we may assume every nonzero projection in A is infinite in B.

There are two cases to be considered, when A'' is a factor and when it is not. Assume first A'' is a factor. Then we can identify B with $A'' \otimes N$, where N is the relative commutant of A in B. By Proposition 6.2 $P = Q \otimes \rho$, where Q is the unique normal positive projection of A'' onto A, and ρ is a normal state on N. If $A = A_h''$, $Q = \iota$, so P is completely positive, hence decomposable. Otherwise, if $R(A)^-$ is the weakly closed real $*$-algebra generated by A, $A'' = R(A)^- + iR(A)^-$ by [16, Thm.2.4] and $Q = \frac{1}{2}(\iota + \alpha)$, where α is the $*$-anti-automorphism $x + iy \to x^* + iy^*$ of A'', $x, y \in R(A)^-$, see proof of [8, Thm.2.2]. Thus Q is decomposable, hence so is P.

Consider next the case when A'' is not a factor. Then P has the form described in Proposition 6.4. Since the expectation Q is completely positive, as noted in Remark 6.5, and the sum of, and composition of, two decomposable maps are decomposable, it suffices to show that e.g. the projection R_e of A_h'' onto A is decomposable. Recall that if x is self-adjoint in A'' then $R_e(x) = R_e(xe) = \bar{x}$, where $\bar{x}$ is the unique operator in A such that $\bar{x}e = xe$. The map $x \to xe$ is clearly decomposable, and the map $\chi : Ae \to Af$ by $ae \to af$ is, as remarked in the proof of Lemma 6.3, a $*$-anti-isomorphism, hence is decomposable. Since $\bar{x} = xe + \chi(xe)$, the map $xe \to \bar{x}$ is decomposable, hence so is R_e. Therefore P is decomposable.

Conversely assume P is a decomposable unital projection of B into itself. Then by Lemma 2.1 P^{**} is a decomposable normal unital projection of B^{**} into itself with $P^{**}(B_h^{**}) = A^-$, where we consider B as contained in B^{**}. As before let r be the support of P^{**}, and recall that $r \in (A+N)'$. Thus P is weakly decomposable, because if $a \in A$ the map $ar \to P^{**}(ar)$ is decomposable, and $P^{**}(ar) = P(a) = a$. In order to show $A+N$ is reversible we shall use Lemma 3.4. For this let π be a $*$-representation of the C^*-algebra C generated by $A+N$ such that $\pi(N) = 0$. Since C is considered as a subalgebra of its second dual we may assume π is normal in order to conclude $\pi(A+N)^-$ is reversible. Since $(A+N)r = Ar$, and $N = \{x \in B_h : xr = 0\}$, there is a normal $*$-representation π' of Cr such that $\pi'(Ar) = \pi(A+N)$. By Lemma 2.5 P^{**} is a decomposable projection of $(rB^{**}r)_h$ onto A^-, and consequently P_r^{**} defined as in Lemma 4.2 by $P_r^{**}(rxr) = P^{**}(rxr)r$ is a decomposable projection of $(B_r^{**})_h$ onto A^-r. Thus by Lemma 2.7 A^-r is reversible, hence $\pi(A+N)^- = \pi'(Ar)^- = \pi'(A^-r)$ is reversible. Since by Lemma 2.8 N is the self-adjoint part of a C^*-algebra, and N is a Jordan ideal in $A+N$ [8, Thm.1.4], it follows from Lemma 3.4 that $A+N$ is a reversible JC-algebra. Q.E.D.

<u>Corollary 7.2</u>. Let B be a unital C^*-algebra and P a unital positive projection of B into itself. Let $A = P(B_h)$ and $N = \{a \in B_h : P(a^2) = 0\}$. Suppose P is faithful when restricted to the C^*-algebra generated by A. Then P is decomposable if and only if $A+N$ is a reversible JC-subalgebra of B_h.

<u>Proof</u>. The assumption that P is faithful on the C^*-algebra (A) generated by A, enforces the support projection r of P^{**} to be separating for (A), hence the map $ra \to a$ extends to a $*$-isomorphism of $(A)r$ onto (A). Thus P is weakly decomposable, and the corollary follows from Theorem 7.1. Q.E.D.

In particular we have when $N = 0$,

Corollary 7.3. Let B be a unital C^*-algebra and P a faithful unital positive projection of B into itself. Then P is decomposable if and only if $P(B_h)$ is a reversible JC-subalgebra of B_h.

The last corollary is also a consequence of

Corollary 7.4. Let B be a unital C^*-algebra and P a unital positive projection of B into itself. Suppose $A = P(B_h)$ is a JC-subalgebra of B_h. Then P is decomposable if and only if A is reversible.

Proof. By Lemma 2.7 if P is decomposable then A is reversible. Conversely if A is reversible the proof of Theorem 7.1 together with Lemma 4.5 shows that P is decomposable. Q.E.D.

It was pointed out in the introduction that if P is a unital positive projection of B into itself then P is completely positive if and only if $P(B)$ is a C^*-algebra with the product $a \cdot b = P(ab)$. The necessity is explicitly proved in [4], while the sufficience does not seem to be explicitly stated in the literature. We therefore include a proof.

Corollary 7.5. Let B be a unital C^*-algebra and P a unital positive projection of B into itself such that $P(B)$ is a C^*-algebra with the product $a \cdot b = P(ab)$. Then P is completely positive.

Proof. Let r be the support of P^{**}. Then by assumption $P(B)r$ is a C^*-subalgebra of B^{**}. In particular the map $ar \to a$ is a $*$-isomorphism of the C^*-algebra $P(B)r$ onto $P(B)$ (so in particular P is weakly decomposable). Since the projection $P_r^{**} : B_r^{**} \to P(B)^{-}r$ is onto a von Neumann subalgebra of B_r^{**}, it is completely positive, [18]. Thus P, being the composition of completely positive maps, is completely positive. Q.E.D.

Corollary 7.6. Let B be a unital C^*-algebra and G a family of Jordan $*$-automorphisms of B. Let $B^G = \{a \in B : \alpha(a) = a$ for all $\alpha \in G\}$, and suppose P is a positive projection of B onto B^G. Then P is decomposable.

Proof. By [14, Cor.3.7], if $a_1, \ldots, a_n \in B^G$ are self-adjoint then

$$\alpha\Big(\prod_1^n a_i + \prod_n^1 a_i\Big) = \prod_1^n \alpha(a_i) + \prod_n^1 \alpha(a_i) = \prod_1^n a_i + \prod_n^1 a_i \in B^G,$$

so B_h^G is a reversible JC-algebra. Thus P is decomposable by Corollary 7.4. Q.E.D.

The next corollary is a dilation theorem for positive linear maps and gives a better understanding of why we need to introduce the rather technical condition that P is weakly decomposable in Theorem 7.1, and why this condition is redundant in the corollaries.

Corollary 7.7. Let C be a unital C^*-algebra and φ a unital positive linear map of C into itself. Let B be the 2×2 matrices over C, and define a map P of B into itself by

$$P((x_{ij})) = \begin{pmatrix} x_{11} & 0 \\ 0 & \varphi(x_{11}) \end{pmatrix}.$$

Then we have

(1) P is a unital positive projection of B into itself.

(2) If $A = P(B_h)$ and $N = \{a \in B_h : P(a^2) = 0\}$ then $A + N = \{(x_{ij}) \in B_h : x_{12} = x_{21} = 0\}$

(3) The following three conditions are equivalent:

(i) P is decomposable.

(ii) P is weakly decomposable.

(iii) φ is decomposable.

<u>Proof</u>. (1). Since $(x_{ij}) \geq 0$ implies $x_{11} \geq 0$, P is positive. Since $\varphi(1) = 1$, $P(1) = 1$. Finally, P is a projection because

$$P(P((x_{ij}))) = P\left(\begin{pmatrix} x_{11} & 0 \\ 0 & \varphi(x_{11}) \end{pmatrix}\right) = \begin{pmatrix} x_{11} & 0 \\ 0 & \varphi(x_{11}) \end{pmatrix}.$$

(2) If $x = (x_{ij}) \geq 0$ then $P(x) = 0$ if and only if $x_{11} = 0$, hence if and only if $x_{11} = x_{12} = x_{21} = 0$, i.e. $N = \{(x_{ij}) \in B_h : x_{11} = x_{12} = x_{21} = 0\}$ Thus (2) follows.

(3) Since $A + N$ is the self-adjoint part of a C^*-algebra, it is in particular a reversible JC-algebra. Thus (i) <=> (ii) by Theorem 7.1. Let e and f denote the projections $\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$ in B. If P is decomposable it follows by Lemma 2.3 and 2.5 that φ is decomposable, because $\varphi(x_{11}) = fP((x_{ij}))f$. Conversely, if φ is decomposable, by Lemma 2.1 $\varphi = \psi + \eta$ where ψ is completely positive and η is completely co-positive. Thus

$$P((x_{ij})) = \begin{pmatrix} x_{11} & 0 \\ 0 & \psi(x_{11}) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \eta(x_{11}) \end{pmatrix},$$

so P, being the sum of a completely positive and a completely co-positive map, is itself decomposable. Q.E.D.

Finally we show a necessary algebraic condition on the eigenspace for the eigenvalue 1 for a positive map in order that it is decomposable.

<u>Corollary 7.8</u>. Let B be a von Neumann algebra and φ a normal

unital positive map of B into itself. Let $B^{\varphi} = \{a \in B_h : \varphi(a) = a\}$, and suppose φ is decomposable. Then B^{φ} has a faithful canonical representation as a reversible JC-algebra.

Proof. Let P be a point-ultraweak limit of the sequence $\{n^{-1} \sum_{0}^{n-1} \varphi^k\}_{n=1,2,\dots}$. Then by [8, Cor.1.6] P is a positive projection of B into itself with $P(B_h) = B^{\varphi}$. By Lemma 2.2 each map $n^{-1} \sum_{0}^{n-1} \varphi^k$ is decomposable, hence by Lemma 2.4. P is decomposable. Let $N = \{a \in B_n : P(a^2) = 0\}$. Then by Theorem 7.1 $B^{\varphi} + N$ is a reversible JC-subalgebra of B_h. If B is considered as a subalgebra of B^{**}, and r is the support projection of P^{**} then the map $B^{\varphi} \to B^{\varphi} r$ is the desired faithful representation of B^{φ} as a reversible JC-algebra. Q.E.D.

References.

1. W. Arveson, Subalgebras of C^*-algebras, Acta math. 123, (1969), 141-224.

2. M. Broise, letter to the author (1967).

3. M.-D. Choi, Positive semi-definite biquadratic forms, Linear Alg. Applic. 12 (1975), 95-100.

4. M.-D. Choi and E. Effros, Injectivity and operator spaces, J.Fnal. Anal. 24 (1974), 156-209.

5. J. Dixmier, Les algèbres d'opérateurs dans l'espaces hilbertien, Paris, Gauthier-Villars (1957).

6. ——————, Les C^*-algèbres et leurs représentations, Paris, Gauthier-Villars (1964).

7. E. Effros and E. Størmer, Jordan algebras of self-adjoint operators, Trans. Amer. Math. Soc. 127 (1967), 313-316.

8. ——————, Positive projections and Jordan structure in operator algebras, Math. Scand. To appear.

9. P. Jordan, J. von Neumann, and E. Wigner, On an algebraic generalization of the quantum mechanical formalism, Ann. Math. 35 (1934), 29-64.

10. R.V. Kadison, A generalized Schwarz inequality and algebraic invariants for operator algebras, Ann. Math. 56 (1952), 494-503.

11. ——————, The trace in finite operator algebras, Proc. Amer. Math. Soc. 12 (1961), 973-977.

12. W.F. Stinespring, Positive functions on C^*-algebras, Proc. Amer. Math. Soc. 6 (1955), 211-216.

13. E. Størmer, Positive linear maps of operator algebras, Acta math. 110 (1963), 233-278.

14. ——————, On the Jordan structure of C^*-algebras, Trans. Amer. Math. Soc. 120 (1965), 438-447.

15. ——————, Jordan algebras of type I, Acta math. 115 (1966), 165-184.

16. ——————, Irreducible Jordan algebras of self-adjoint operators, Trans. Amer. Math. Soc. 130 (1968), 153-166.

17. ——————, Positive linear maps of C^*-algebras, Lecture Notes in Physics 29, 85-106, Berlin-Heidelberg-New York; Springer, 1974.

18. J. Tomiyama, On the projection of norm one in W^*-algebras, Proc. Japan Acad. 33 (1957), 608-612.

19 D. Topping, Jordan algebras of self-adjoint operators, Mem. Amer. Math. Soc. 53 (1965).

20. S.L. Woronowics, Positive maps of low dimensional matrix algebras, Rep. Math. Phys. 10 (1976), 165-183.

21. ———, Nonextendible positive maps, Commun. Math. Phys. 51 (1976), 243-282.